Indian Statistical Institute, Bangalore
M. Math. First Year
First Semester - Linear Algebra
Back Paper Exam Duration : 3 hours Max Marks 100 Date : January 03, 2018

1. Define the wedge powers of a vector space, and prove their existence. [20]

2. (a) If $S, T : V \to V$ are commuting linear operators on a vector space then show that if W is an invariant subspace for T then $S(W)$ is also T-invariant.

   (b) Hence, show that if $T : V \to V$ is a normal operator on an inner product space and W is a T-invariant subspace then so is $W^{\perp}$. [10 + 10 = 20]

3. (a) If G is a non-trivial finite abelian group then show that there are numbers $1 < d_1 \mid d_2 \mid \ldots \mid d_n$ such that G is isomorphic to the direct product of the groups $\mathbb{Z}/d_i\mathbb{Z}, 1 \leq i \leq n$.

   (b) Show that the numbers $d_i, 1 \leq i \leq n$ in (a) are uniquely determined by G. [10 + 10 = 20]

4. (a) Let V_n be the vector space of all polynomials of degree $< n$ (over some field). Define $T : V_n \to V_n$ by $(Tf)(x) = \frac{f(x)-f(0)}{x}$. Find all the eigenvalues and eigenspaces of T.

   (b) Let $E : V \to V$ be a linear operator on a vector space such that $E^2 = E$. Then find all the eigenvalues and eigenspaces of E. [10 + 10 = 20]

5. (a) Show that all the submodules of the $\mathbb{Z}$-module $\mathbb{Z}^n$ are free of rank $\leq n$.

   (b) For each $n \geq 1$, given an example of a proper submodule of $\mathbb{Z}^n$ of rank n. [10 + 10 = 20]